CRC Handbook of **Tables for
Order Statistics from
Inverse Gaussian
Distributions
with Applications**

N. Balakrishnan
William W.S. Chen

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N. Balakrishnan

Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario, Canada

William W.S. Chen

Internal Revenue Service
Washington, DC

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To Sudha, Shubha, Ravi and Aditya
To Heng Chen

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Preface

Inverse Gaussian distributions have recently become one of the most commonly studied models in the theoretical literature while being commonly used in the applied literature. Though these distributions have been around for over fifty years, much of the work concerning inferential methods for the parameters of inverse Gaussian distributions has been carried out only in the recent past. Most of these methods, particularly those based on censored samples, do involve the use of numerical methods to solve some equations.

Order statistics and their moments have been studied quite extensively for numerous distributions. However, moments of order statistics from inverse Gaussian distributions have not been discussed in the statistical literature until now primarily due to the extreme computational complexity in their numerical determination. It is this challenge that motivated us to enter into this tremendous task of computing the means, variances and covariances of order statistics for all sample sizes up to twenty five and for many choices of the shape parameter. Another major reason for taking up this study is, of course, to make use of the tabulated values of means, variances and covariances of order statistics in order to derive the best linear unbiased estimators of the location and scale parameters based on complete as well as Type-II censored samples. Both these goals have been achieved successfully and the extensive tables that emerged out of our efforts have been presented in this volume. We have also presented a few examples to illustrate some practical applications of these tables.

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N. BALAKRISHNAN
Hamilton, Ontario, Canada

W. W. S. CHEN
Washington, DC

May 1997

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Introduction

Inverse Gaussian distributions have recently become one of the commonly used statistical models in various problems. Much of this is due to pioneering contributions made by numerous researchers; in particular, the early work by Halphen¹, Tweedie, Wald and Wasan, and more recent work by Chhikara, Folks, Seshadri, Whitmore, Cohen and Jørgensen. Due to the extensive developments made on the Inverse Gaussian distributions during the past thirty years or so, this wide family of distributions started getting due attention from theoreticians as well as applied researchers. As a result, numerous theoretically sophisticated methodologies have been developed based on the Inverse Gaussian distributions. Simultaneously, the Inverse Gaussian distributions were also being used as models for analyzing data in a variety of applied problems.

Tweedie (1947) first noted the inverse relationship between the cumulant generating functions of a wide class of distributions and those of Gaussian (normal) distributions and consequently termed that class of distributions as **Inverse Gaussian distributions**. Independently, Wald (1947) derived the same class of distributions as an asymptotic form of distributions of average sample number (ASN) in sequential analysis. As a result, the name **Wald distributions** is also used for some members of the class of Inverse Gaussian distributions. Wasan (1968) observed the Inverse Gaussian distributions while studying the first passage time of Brownian motion and hence called them **first passage time distributions of Brownian motion with positive shift**, and denoted them by *T.B.M.P. distributions*.

Due to the prominence and wide applicability that the Inverse Gaussian distributions have achieved over the years and especially during the recent times, two books have been devoted completely to the theory and applications of Inverse Gaussian distributions; see Chhikara and Folks (1989) and Seshadri (1993). For a concise review of developments on this class of distributions, one may also refer to Chapter 15 of Johnson, Kotz and Balakrishnan (1994). Koziol (1988) has prepared a handbook of tables giving percentile points of Inverse Gaussian distributions.

As mentioned earlier, the class of Inverse Gaussian distributions is currently being used quite commonly as a life-time model in reliability studies. Because of this particular application, several authors have paid special attention to inferential issues concerning these distributions. Both the two-parameter (scale and shape parameters) and the three-parameter (location, scale and shape parameters) models have been discussed in this context. The books by Chhikara and Folks (1989) and Seshadri (1993) present extensive discussions on classical inference for the parameters of Inverse Gaussian distributions. Some simple and efficient methods of estimation and their relative merits have been

¹Thanks are due to Professor V. Seshadri for bringing to our attention some historical facts about the original work of Etienne Halphen on the Inverse Gaussian and generalized Inverse Gaussian distributions.

discussed in the books by Cohen and Whitten (1988) and Balakrishnan and Cohen (1991).

In this book, we are primarily concerned with the three-parameter form of Inverse Gaussian distributions although the tables of moments of order statistics included here can be easily used to discuss the estimation for the two-parameter form as well. As can be noted from the books by Cohen and Whitten (1988) and Balakrishnan and Cohen (1991), the estimation problem associated with the three-parameter Inverse Gaussian distribution is a difficult and challenging one. In this connection, we have undertaken in this book a two-fold study - firstly to study order statistics from standard Inverse Gaussian distributions and their moments, and secondly to apply the computed moments of order statistics in order to derive the best linear unbiased estimators of the location and scale parameters (with the shape parameter being fixed) based on complete as well as Type-II censored samples. In order to achieve these goals, we had to resolve several numerical complications and also employ many checks in order to verify the accuracy of the computations performed. These will be explained in Chapter 3 along with other pertinent details concerning the computations carried out for this book.

In Chapter 2, we start with a brief presentation of different forms of Inverse Gaussian distributions and some basic properties. We then describe a reparametrization suggested by Chan, Cohen and Whitten (1983) using which a three-parameter form of Inverse Gaussian distributions is presented. Plots of the standard form of this density function are also presented for various values of the shape parameter along with an indication of its departure from the standard normal density function.

In Chapter 3, we start with an introduction to order statistics and then present the necessary formulas for the marginal and joint densities of order statistics and the computational formulas for the means, variances and covariances of order statistics. Next, we describe the computational procedure used for the determination of the means, variances and covariances of order statistics and explain some of the numerical complications that had to be resolved while doing these computations. We also present the relations and identities that we used for checking the accuracy of the computations, and demonstrate them with a few examples. Values of means of order statistics for sample sizes up to twenty five and for values of the shape parameter being 0.0(0.1)2.5 computed by a numerical (single) integration procedure are presented in Table 1. Similarly, values of variances of order statistics for sample sizes up to twenty five and for values of the shape parameter being 0.0(0.1)2.5 computed by a numerical (single) integration procedure are presented in Table 2. Finally, values of covariances of order statistics for sample sizes up to twenty five and for values of the shape parameter being 0.0(0.1)2.5 computed by a numerical (double) integration procedure are presented in Table 3.

One of the primary applications of means, variances and covariances of order statistics is in deriving best linear unbiased estimators of the location and scale parameters; see, for example, David (1981), Balakrishnan and Cohen (1991), and Arnold, Balakrishnan and Nagaraja (1992). In Chapter 4, we first present the necessary formulas for the best linear unbiased estimators and then describe the computational procedure employed for the derivation of these best linear unbiased estimators. We discuss the best linear unbiased estimation of the location and scale parameters (assuming the shape parameter to be fixed) based on complete as well as Type-II right-censored samples which are most likely to be encountered in life-testing experiments. The coefficients in the best linear unbiased estimator of the location parameter for sample sizes up to twenty five, for values of the shape parameter being 0.0(0.1)2.5, and for various levels of censoring are presented in Table 4. Similarly, the coefficients in the best linear unbiased estimator of the scale parameter for sample sizes up to twenty five, for values of the shape parameter being 0.0(0.1)2.5, and for various levels of censoring are presented in Table 5. Finally, the

variances and covariance of these best linear unbiased estimators for sample sizes up to twenty five, for values of the shape parameter being 0.0(0.1)2.5, and for various levels of censoring are presented in Table 6.

In Chapter 5, we present some numerical examples in order to illustrate a direct application of these tables. The best linear unbiased estimators so derived can also be used successfully to develop best linear unbiased prediction of the failure times corresponding to the censored units. This is demonstrated in Chapter 6 and is also illustrated with some numerical examples. Another very useful application of means of order statistics is in constructing Q-Q plots and in developing formal correlation-type goodness-of-fit tests in order to verify the validity [see D'Agostino and Stephens (1986)] of Inverse Gaussian distributions for the observed data. In Chapter 7, we illustrate this use of means of order statistics presented in Table 1 through the numerical examples considered earlier. Through Monte Carlo simulations, we also demonstrate the correlation-type goodness-of-fit test with the help of the previously considered numerical examples.

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