Nonlinear Reaction-Diffusion-Convection Equations

Lie and Conditional Symmetry,
Exact Solutions, and
Their Applications

Roman Cherniha
Mykola Serov
Oleksii Pliukhin

A CHAPMAN & HALL BOOK

MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

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Preface

Second-order partial differential equations (PDEs) have played a crucial role in mathematical modeling a wide range of processes in natural and life sciences since the 18th century. Typically linear PDEs are used in order to describe various processes, while *nonlinear partial differential equations* have been widely involved for such purposes only since the beginning of the 20th century. Nowadays it is generally accepted that a huge number of real world processes arising in physics, biology, chemistry, material sciences, engineering, ecology, economics etc. can be adequately described only by nonlinear PDEs.

At the present time, there is no existing general theory for integration of nonlinear PDEs, hence construction of particular exact solutions for these equations remains an important mathematical problem. Finding exact solutions that have a clear interpretation for the given process is of fundamental importance. In contrast to linear PDEs, the well-known principle of linear superposition cannot be applied to generate new exact solutions for nonlinear PDEs. Thus, the classical methods for solving linear PDEs are not applicable to nonlinear PDEs. Of course, a change of variables can sometimes be found that transforms a nonlinear PDE into a linear equation (the classical example is the Cole–Hopf substitution for the Burgers equation). It was stated by W.F. Ames in 1965 that “transformations are perhaps the most powerful general analytic tool currently available in this area”. However, finding exact solutions of a large majority of nonlinear PDEs requires new methods. Nowadays, 50 years later, the most powerful methods for construction of exact solutions to nonlinear PDEs are the symmetry-based methods, in particular the Lie method and the method of nonclassical (i.e., non-Lie) symmetries.

The Lie method (the terminology “the Lie symmetry analysis” and “the group analysis” are also used) is based on finding Lie’s symmetries of a given PDE and using the symmetries obtained for the construction of exact solutions. The method was created by the prominent Norwegian mathematician Sophus Lie in the 1880s. It should be pointed out that Lie’s works on application Lie groups for solving PDEs were almost forgotten during the first half of the 20th century. In the end of the 1950s, L.V. Ovsiannikov inspired by Birkhoff’s works devoted to application of Lie groups in hydrodynamics, rewrote Lie’s theory using modern mathematical language and published a monograph in 1962, which was the first book (after Lie’s works) devoted fully to this subject. The Lie method was essentially developed by L.V. Ovsiannikov, W.F. Ames, G. Bluman, W.I. Fushchych, N. Ibragimov, P. Olver, and

other researchers in the 1960s–1980s. Several excellent textbooks devoted to the Lie method were published during the last 30 years, therefore one may claim that it is the well-established theory at the present time. Notwithstanding the method still attracts the attention of many researchers and new results are published on a regular basis. In particular, solving the so-called problem of group classification (Lie symmetry classification) still remains a highly non-trivial task and such problems are not solved for several classes of PDEs arising in real world applications.

On the other hand, it is well-known that some nonlinear RDC equations arising in applications have a “poor” Lie symmetry. For example, the Fisher and Fitzhugh–Nagumo equations, which are widely used in mathematical biology, are invariant only under the time and space translations. The Lie method is not efficient for such equations since it enables one to construct only those exact solutions, which can be obtained without using this cumbersome algorithm. Taking into account this fact, one needs to apply other approaches for solving such equations. The best known among them is the method of nonclassical symmetries proposed by G. Bluman and J. Cole in 1969. Although this approach was suggested almost 50 years ago its successful applications for solving nonlinear equations were accomplished only in the 1990s owing to D.J. Arrigo, P. Broadbridge, P. Clarkson, J.M. Hill, E.L. Mansfield, M.C. Nucci, P. Olver, E. Pucci, G. Saccomandi, E.M. Vorob’ev, P. Winternitz and others. A prominent role in applications and further development of the nonclassical symmetry method belongs to the Ukrainian school of symmetry analysis, which was created in the early 1980s and led by W.I. Fushchych (V.I. Fyshchich) until 1997 when he passed away. In particular, a concept of conditional symmetry was worked out and its applications to a wide range of nonlinear PDEs were realized by M. Serov, I. Tsyfra, R. Zhdanov, R. Popovych, R. Cherniha and others. Notably, following Fushchych’s proposal dating back to 1988, we continuously use the terminology “ Q -conditional symmetry” instead of “nonclassical symmetry”.

We also note that several other approaches for solving nonlinear PDEs (in particular, evolution equations) were independently suggested in the 1990s–2000s. Not pretending to completeness and precise statement, the following of them should be mentioned: the method of linear invariant subspaces, the method of generalized conditional symmetries, the method of heir-equations, the method of linear determining equations, the method of additional generating conditions etc. Notwithstanding some of these methods formally do not use any symmetries, a deep analysis shows that they are related to symmetry-based methods.

The main mathematical object of this book is the class of nonlinear reaction-diffusion-convection equations (RDC). In our opinion, nonlinear RDC equations possess the most important role among other nonlinear equations. One cannot imagine a correct mathematical model describing heat and mass transfer, filtration of liquid, solute transport in tissue, diffusion in chemical reactions, tumor growth and many other processes without RDC equations. The

importance of RDC equations in real world applications follows from the fact that they model three main transport mechanisms: diffusion (heat transfer), reaction (source/sink), and convection (advection). Thus, they have been extensively studied by means of different mathematical methods and techniques, including symmetry-based methods.

This book is devoted to (i) search Lie and Q -conditional (non-classical) symmetries of nonlinear RDC equations; (ii) constructing exact solutions using the symmetries obtained and using the method of additional generating conditions; (iii) applications of the solutions derived for solving some biologically and physically motivated problems.

The monograph summarizes in a unique way the results derived by the authors during the last 20 years. Notably, the first joint paper was written by R. Cherniha and M. Serov about 20 years ago during our stay at the Mathematisches Forschungsinstitut Oberwolfach (Germany), while the last joint paper of M. Serov and O. Pliukhin was published in 2015. It should be pointed out that a number of misprints, inexactness and even mistakes arising in our papers were corrected during the book preparation, and new unpublished results were included in Chapters 3, 4 and 5. Moreover, our results are supplemented by those obtained by other authors. As a result, the reader will realize a huge progress, which has been done in study of nonlinear RDC equations by means of symmetry-based methods since the 1990s.

The book presents a most complete (at the present time) description of Lie and conditional symmetries for nonlinear RDC equations, which are very common in real world applications. The most interesting subclasses from this class (like equations with power-law and exponential nonlinearities) are extensively studied. In particular, an essential stress is made on finding symmetries and exact solutions for the widely used equations in bio-medical applications, including Fisher, Murray, Fitzhugh-Nagumo and Kolmogorov-Petrovskii-Piskunov type equations. Concerning the equations listed above and their generalization, a number of examples are presented, in which the relevant real world models are analytically solved, and a biological/physical interpretation of the solutions obtained is provided.

In Introduction (Chapter 1), some mathematical models based on nonlinear RDC equations are discussed, methods for constructing exact solutions of nonlinear PDEs are briefly presented together with a short historical review. The remaining part of this chapter is devoted to the main notions, definitions, and theorems, which form theoretical background of the Lie method and other symmetry-based methods.

Chapter 2 is partly devoted to the linear and nonlinear diffusion (heat) equations, including the multi-dimensional case. Here we present the well-known results of the Lie symmetry classification (the group classification) together with some applications for constructing exact solutions. The main part of Chapter 2 is devoted to the complete Lie symmetry classifications (LSC) of the general class of RDC equations

$$u_t = [A(u)u_x]_x + B(u)u_x + C(u),$$

where $u = u(t, x)$ is an unknown function, $A(u), B(u), C(u)$ are arbitrary smooth functions and the subscripts t and x denote differentiation with respect to these variables. First, LSC is derived using the well-known Lie-Ovsiannikov approach based on the equivalence transformations. Afterwards the second LSC is obtained via so-called form-preserving transformations. An extensive discussion including nontrivial examples is presented to show advantages of application of the form-preserving transformations in order to solve the LSC problem.

Chapter 3 is devoted non-Lie symmetries of nonlinear PDEs. First, we present a historical review concerning conditional symmetry of PDEs, introduce notion of Q -conditional symmetry (nonclassical symmetry) and repeat some well-known results about non-Lie symmetry of nonlinear diffusion equations. The main part of Chapter 3 is devoted to the Q -conditional symmetry classifications of the general class of RDC equations. In contrast to the LSC problem, the result is incomplete, however, a complete classification is derived for several important (from applicability point of view) subclasses of the general class. Probably the most important among them is the Burgers type equations of the form

$$u_t = u_{xx} + \lambda uu_x + C(u), \quad \lambda \in \mathbb{R}.$$

In fact, the above class of equations contains as particular cases the Fisher, Murray, and Fitzhugh–Nagumo equations and their natural generalizations used widely in modeling of biomedical and ecological processes.

Chapter 4 is fully devoted to construction of exact solutions. A wide range of RDC equations are examined in order to search for both Lie and non-Lie solutions. Several examples are presented, which show how nontrivial exact solutions can be constructed for some well-known nonlinear equations. In particular, our attention is addressed to the nonlinear RDC equations with constant and power-law diffusivities, arising in bio-medical and ecological applications. For such equations, we construct exact solutions, examine their properties and (in some cases) provide their biological interpretation. The RDC equations with exponential nonlinearities are also under study in this section.

Chapter 5 is devoted to the method of additional generating conditions and its application for solving some nonlinear RDC equations. This method can be treated as a particular case of the method of differential constraints. Basic ideas of the method of differential constraints have roots in Darboux's works. In the 1960s, N.N. Yanenko formalized the method using the modern mathematical language. However, he has not provided any constructive algorithm for finding the compatible differential constraints for a PDE in question. The method of additional generating conditions solves this problem for PDEs, which can be reduced to those with quadratic nonlinearities. The chapter contains a detailed description of the method, examples demonstrating its efficiency in the case of the nonlinear RDC equations with power-law and ex-

ponential nonlinearities. An extensive comparison of the solutions obtained with those derived via other techniques is also presented.

Chapters 2, 3 and 4, which form the main part of this monograph and are essentially connected each with another, were written by all the authors and they contributed equally to these chapters.

Chapters 1 and 5 were written by R. Cherniha.

The book is a monograph. Its academic level suits graduate students and higher. Some parts of the book may be used in “Mathematical Biology” and “Nonlinear Partial Differential Equations” courses for master students and in the final year of undergraduate studies. Nowadays such courses are common in all leading universities over the world.

The book was typeset in LaTeX using the CRC Press templates, the figures were drawn using the computer algebra package **Maple** and some calculations were done using **Mathematica**.

Last but not the least, we are grateful to our colleagues and our teacher Wilhelm Fushchych (1936–1997), who was the supervisor for R. Cherniha and M. Serov in the 1980s. This book could not have been written without his innovative scientific ideas and many years of his support. The authors thank their Ukrainian colleagues for fruitful discussions, valuable critique, and helpful suggestions, which helped us to write this modest work. Especially, we are grateful to Vasyl’ Davydovych, Sergii Kovalenko, Inna Rassokha, and Valentyn Tychynin.

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Acronyms

DE(s) – determining equation(s)

Eq(s). – equation(s)

ET(s) – equivalence transformation(s)

FN – Fitzhugh–Nagumo

FPT(s) – form-preserving transformation(s)

KPP – Kolmogorov–Petrovskii–Piskunov

LSC – Lie symmetry classification

MAGC – method of additional generating conditions

MAI(s) – maximal algebra(s) of invariance

MGI – maximal group of invariance

ODE – ordinary differential equation

QSC – Q -conditional symmetry classification

PDE – partial differential equation

RD – reaction-diffusion

RDC – reaction-diffusion-convection

w.r.t. – with respect to

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Chapter 1

Introduction

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1.1 Nonlinear reaction-diffusion-convection equations in mathematical modeling

Since the 17th century when G. Leibnitz and I. Newton discovered differential and integral calculus, differential equations are the most powerful tools for mathematical modeling various processes in physics, chemistry, biology, medicine, ecology, economics etc. Of course, pioneering models were created in order to express some classical laws (like the second Newton law) in physics and astronomy, later differential equations came to be used for describing a wide range of processes not only in physics but also in other natural and life sciences. It can be noted that almost all mathematical models created before the end of the 19th century were based on ordinary differential equations (ODEs) and linear partial differential equations (PDEs). *Nonlinear PDEs* are widely used in mathematical modeling real world processes since the beginning of the 20th century only. Probably, one of the first attempts in applying and solving a nonlinear PDE of the parabolic type was made by J. Boussinesq who studied the porous diffusion equation describing the water filtration in soil [32]. One of the first applications of nonlinear reaction-diffusion (RD) equations in biology was proposed by R.A. Fisher [98, 99].

Nowadays, i.e., 100 years later, it is generally accepted that a huge number of real processes arising in physics, biology, chemistry, material sciences, engineering, ecology, economics etc. can be adequately described only by *nonlinear PDEs* (or systems of such equations). The most widely used type of equations for modeling such processes are the nonlinear reaction-diffusion-convection (RDC) (advection) equations. Since 1952 when A.C. Turing published the remarkable paper [240], in which he proposed a revolutionary idea about mechanism of morphogenesis (the development of structures in an or-

ganism during the life), nonlinear RD equations (including those with convective terms) play a crucial role in real world applications and have been extensively studied by means of different mathematical methods/techniques. As a result, in the 1970s several monographs were published, which are devoted to study and application of the nonlinear reaction-diffusion-convection (RDC) equations in physics [5, 6, 158], biology [97, 179] and chemistry [10, 11]. In our opinion, these books had a great impact attracting many scholars to study the nonlinear RDC equations and use them for modeling real world processes. Since that time many other excellent monographs and textbooks appeared, especially for models related to life sciences (see, e.g., [35, 96, 160, 181, 182, 194, 231, 245]). We concentrate ourselves mostly on biologically motivated models in what follows.

Typically the RDC equation describing a process in the 1D space approximation has the form

$$u_t = [A(u)u_x]_x + B(u)u_x + C(u),$$

where A, B and C are some given functions, while $u(t, x)$ is an unknown function (hereafter the lower indices t and x denote differentiation with respect to (w.r.t.) these variables). In the models related to biomedical applications, the function $u(t, x)$ means the concentration of cells (population, drugs, molecules). The functions A, B and C are related to the three most common types of transport mechanisms occurring in real world processes. The diffusivity $A > 0$ (typically it is a constant) is the main characteristic of the diffusion process, the term $B(u)u_x$ (B typically means velocity, which can be positive and/or negative) describes the convective transport (in contrast to diffusion, one is not random) and the reaction term $C(u)$ describes the process kinetics (for example, this function presents interaction of the population u with the environment). A natural multidimensional analog of the above equation reads as

$$u_t = \nabla \cdot (A(u)\nabla u) + V(u) \cdot \nabla u + C(u). \quad (1.1)$$

Here u is the function of t and $x_1, \dots, x_n$, $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, the velocity $B(u)$ is replaced by the velocity vector $V(u)$ and $\cdot$ means the scalar product.

In (1.1) the variable diffusivity $D(u)$ (typically it is a power-law function but can be the function with a more complicated structure [40, 160]) arises in more and more modeling situations of biomedical importance from diffusion of genetically engineered organisms in heterogeneous environments to the effect of white and grey matter in the growth and spread of brain tumors [160, 182]. For example, the power-law diffusivity occurs as an extension of the classical diffusion model, when there is an increase in diffusion due to population pressure (see Section 11.3 in [182] and references therein).

The velocity vector V can be a vector function depending on the concentration u (if $V = \text{const}$ then the term $V \cdot \nabla u$ is removable from (1.1) by the Galilei transformations) and the simplest case when V is linear w.r.t. u was firstly studied in [179]. Notably, the velocity vector typically has the structure

$V = (B(u), 0 \dots, 0)$ in real world models (for example, the evolution of fish population in a river can be adequately described if one takes into account the stream velocity only in the direction x_1 and neglects in other directions).

Although the reaction term C can possess a great variety of forms depending on the model in question (see examples in the books cited above), the most typical form of the reaction term is $C(u) = \lambda_1 u^p - \lambda_2 u^q$ with the positive exponents p and q (see Sections 11.3 and 13.4 in [182]). Depending on values of p and q several well-known equations arising in biomedical applications can be identified. For example, setting $p = 1$, $q = 2$, $A(u) = d = \text{const}$ and $V = 0$, the famous Fisher equation

$$u_t = d\Delta u + u(1 - u),$$

(here Δ is the Laplace operator) is obtained describing the spread in space of a favored gene in a population. Setting $p = 2$ and $q = 3$, the Huxley equation [43, 88]

$$u_t = d\Delta u + u^2(1 - u)$$

is derived, which can be thought as a limiting case of the famous Fitzhugh–Nagumo (FN) equation [100, 185]

$$u_t = d\Delta u + u(u - \delta)(1 - u), \quad 0 < \delta < 1.$$

The latter is a simplification of the celebrated Hodgkin–Huxley model [133] describing the ionic current flows for axonal membranes. The FN equation reduces the Hodgkin–Huxley model, which has a very complex structure, and describes the nerve impulse propagation. The function $u(t, x)$ means the electric potential across the cell membrane.

In conclusion, we note that there are real world processes, which are described by the RDC equations involving coefficients A, B and C depending on derivatives of the function u (see, e.g., the recent paper [69] and works cited therein). Examination of such equations lies beyond the scope of this monograph.

1.2 Main methods for exact solving nonlinear reaction-diffusion-convection equations

As it was already pointed out, it is a generally accepted fact at the present time that a huge number of real processes arising in physics, biology, chemistry etc. can be adequately described by *nonlinear* partial differential equations (PDEs) only. On the other hand, the well-known principle of linear superposition cannot be applied to generate new exact solutions to nonlinear PDEs. Thus, the classical methods (the Fourier method, the Green function

method, the method of the Laplace transformations, and so forth) are not applicable for solving nonlinear equations.

At the present time, there are many methods/techniques, which allow us to construct particular solutions of some nonlinear PDEs, however those are applicable to correctly-specified classes of PDEs only and any general integration theory is unknown. While there is no existing general theory for integrating nonlinear PDEs, construction of particular exact solutions for these equations is *a nontrivial and important problem*. Finding exact solutions that have a physical, chemical or biological interpretation is of *fundamental importance*.

One may say that the oldest technique for solving nonlinear differential equation is finding an appropriate transformation for a given PDE. In fact, a change of variables can sometimes be found that transforms the given nonlinear PDE into a linear equation. Transformation of the Burgers equation into the linear heat equations via the Cole–Hopf substitution is the classical example in this direction. However, finding exact solutions of most nonlinear PDEs generally requires other methods/techniques than those for linear equations.

Nowadays the most powerful methods for construction of exact solutions for a wide range of classes of nonlinear PDEs are symmetry-based methods. All these methods have the common idea stating that exact solutions (at least particular ones) can be found for a given PDE provided its symmetry (a set of symmetries) is known. These methods originated from the Lie method, which was created by the prominent Norwegian mathematician Sophus Lie in the end of 19th century [166, 167] (see also the reprints in [168, 169]). The method was essentially developed using modern mathematical language by L.V. Ovsiannikov, G. Bluman, N. Ibragimov, W.F. Ames and some other researchers in the 1960s–1970s. Although the technique of the Lie method is well-known, the method still attracts attention of researchers and new results are published on a regular basis. In the next section a short historical review, basic notions, examples and theorems of the Lie method are presented.

In 1969, G. Bluman and J. Cole introduced an essential generalization of the Lie symmetry notion [26], which later was called nonclassical symmetry (in order to distinguish the new kind of symmetry from the classical Lie symmetry). Although nonclassical symmetries were not used for examination of nonlinear PDEs almost for 20 years (until the late 1980s), nowadays it is a powerful tool for constructing exact solutions of nonlinear equations. In particular, many important results for evolution PDEs were obtained during the last two decades. In the first section of Chapter 3, a short historical review, basic notions and theorems of the nonclassical method are presented. Notably, the terminology “nonclassical symmetry” is not generally accepted because notions “ Q -conditional symmetry” and “reduction operator” are widely used too (see discussion on this matter in Section 3.1).

In the 1980s–1990s, a few new types of symmetries were introduced, which also allow us to construct exact solutions of nonlinear PDEs. The notion of conditional symmetry was suggested by Fushchych and his collaborators [112], [114, Section 5.7]. Note that the notion of nonclassical symmetry can be

derived as a particular case from conditional symmetry but not vice versa (see highly nontrivial examples in Section 3.1 and in paper [63, 64]). Weak symmetry was suggested in [198, 199], potential symmetry was introduced in [29, 30, 159], while generalized conditional symmetry was independently formulated in [101, 173] and [251] (the terminology “conditional Lie–Bäcklund symmetry” was used in the latter).

The crucial idea used for introducing new types of symmetries can be formulated as follows. Let us consider an arbitrary PDE. For simplicity we restrict ourselves to the second-order two-dimensional equation

$$L(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0. \quad (1.2)$$

Hereafter $u = u(t, x)$ is an unknown smooth function¹, while L is a given smooth function. Almost all known in literature symmetries of Eq. (1.2) can be written as the differential operator

$$X = \xi^0 \partial_t + \xi^1 \partial_x + \eta \partial_u, \quad (1.3)$$

where the coefficients ξ^0, ξ^1 and η should be found according to the definition (criteria) of symmetry in question. In the case of the Lie, nonclassical and conditional symmetries, the coefficients depend on dependent and independent variables at maximum (those can be simply constants) and the corresponding criteria for their finding are presented in Sections 1.3 and 3.1.

However, the operator X has a more complicated structure if one is looking for other types of symmetries. For example, the coefficient η depends on derivatives of the function u in the case of generalized conditional symmetries [101, 173, 220].

In the case of potential symmetries, the coefficients depend on integrals of u , so that nonlocal operators are obtained. Thus, the terminology “nonlocal symmetry” is also used (see, e.g., [241] and references therein). A substantial number of examples involving potential symmetries for examination of nonlinear PDEs is presented in [25] (see also references therein).

A vast literature (see, e.g., [196] and citations therein) is devoted to higher-order symmetries (the terminology “generalized” and “Lie–Bäcklund” symmetry is also used) of the form

$$Z = X + \zeta^0 \partial_{u_t} + \zeta^1 \partial_{u_x} + \zeta^{11} \partial_{u_{xx}} + \zeta^{01} \partial_{u_{tx}} + \dots, \quad (1.4)$$

where coefficients depend on derivatives of the function u . Such symmetries were introduced by Noether in her remarkable work [190]. Here we want only to stress that integrability of nonlinear PDEs via the method of inverse scattering problem [1, 93] is related with higher-order symmetries and conservation laws [177].

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¹Throughout the book the notion “smooth function” means that one is differentiable with respect to (w.r.t.) its variables up to the equation order, i.e., in the case of Eq. (1.2), u is the twice differentiable function w.r.t. t and x (at least in an open domain)

At the present time, it is a widely accepted hypothesis that each known exact solution of a given nonlinear PDE can be derived using an appropriate symmetry. On the other hand, there are some efficient methods/techniques allowing to construct exact solutions without knowledge of any symmetry. Here we are not going to present all of them because nowadays there are too many techniques proposed by a huge number of authors and it is very difficult to classify them (notably some new methods are particular cases of those proposed in the 1990s and earlier). In our opinion, the most general approach called the method of differential constraints was formulated in the 1960s [249] and was further developed in monograph [232]. Actually, basic ideas of the method of differential constraints have roots in Darboux's works [103] (see also excellent historical reviews on this matter in [214] and [119]). The main idea of the method of differential constraints is very simple: to define suitable constraint(s) for a given PDE in such a way that the overdetermined system obtained will be compatible and can be (partly) solved using the existing methods. Methods for solving overdetermined systems of differential equations were known since the first half of the 20th century [42, 139] and can be successfully applied in many cases (for instance, see examples in [232]). However, the main problem of the method is how to define the suitable constraint(s). At the present time, the corresponding algorithm does not exist in general case and one may claim that the method of differential constraints is rather a fruitful idea without a constructive algorithm. Interestingly, the symmetry-based methods implicitly use this idea in a constructive way. In fact, in order to find exact solutions, one solves the given nonlinear PDE (system of PDEs) together with the differential constraint(s) generated by a symmetry operator.

There are several techniques, which propose to use the correctly-specified differential constraints in order to find exact solutions for some correctly-specified classes of PDEs. In particular, the method of additional generating conditions [45, 46, 48] and the method of determining equations [143, 144] were independently worked out in the 1990s. Both methods are very similar and Chapter 5 is devoted to the first of them.

Several approaches based on substitutions of the special form, which are often called *ansatz*², should be mentioned. Such substitution reduces the given PDE to a simpler equation (e.g., ODE) or a system of simpler equations, which can be integrated (at least partly). In order to construct the relevant *ansatz*, either some physical (biological, chemical etc.) motivation or an ad hoc approach are used. The most typical is the plane wave *ansatz*

$$u = \phi(\omega), \quad \omega = x - vt, \quad v \in \mathbb{R}, \quad (1.5)$$

which reduces Eq. (1.2) to an ODE provided the function L does not depend on t and x . Although solving the nonlinear ODE obtained can be also a non-trivial problem, the solution (at least partial) can be usually found by using the classical methods or handbooks like [142, 212]. Notably, there exist some

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²Ansatz (pl. ansätze) is a German word

recent techniques (e.g., the tanh- and exp-function methods) allowing to construct such solutions of the nonlinear ODE obtained, which lead to traveling waves (fronts) of the given PDE (see, e.g., [174, 175, 246]), however their wide applicability is often questionable [161].

The second classical ansatz follows from the Furrier method (the method of separation of variables) and reads as

$$u = T(t)X(x), \quad (1.6)$$

where $T(t)$ and $X(x)$ are to-be-determined functions. Depending on the form of L , PDE (1.2) can be reduced to a system of two ODEs. Both ansätze are well-known since the 19th century and related with Lie symmetry of the equation in question. The first one reflects invariance under the time and space translations, the second is related to scale invariance of the given equation. Notably, J. Boussinesq was the first to construct an exact solution of the nonlinear physically motivated problem using ansatz (1.6) [32] (see Section 4.2 for details).

Several new ansätze were suggested during the recent decades in order to construct exact solutions for nonlinear PDEs, especially evolution equations. Many of them can be united and written in the form of the ansatz with separated variables

$$A(u) = \varphi_0(t)g_0(x) + \dots + \varphi_{m-1}(t)g_{m-1}(x). \quad (1.7)$$

Of course, (1.7) is the straightforward generalization of ansatz (1.6). Here some functions are known, others should be found. Typically, the functions $\varphi_0, \dots, \varphi_{m-1}(t)$ are unknown in the case of evolution equations, the functions $g_0(x), \dots, g_{m-1}(x)$ are specified by either physically motivated reasons or an ad hoc approach, while $A(u)$ reflects usually nonlinearities in the given PDE. As a result, a m -dimensional system of ODEs should be obtained for finding $\varphi_0, \dots, \varphi_{m-1}(t)$ by using ansatz (1.7). For example, ansatz (1.7) with the power-law functions $g_k = x^k$, i.e.,

$$A_0(U) = \varphi_0(t) + \varphi_1(t)x + \dots + \varphi_{m-1}(t)x^{m-1}, \quad (1.8)$$

was extensively used for constructing exact solutions of reaction-diffusion (RD) equations with power and exponential nonlinearities in the 1990s [117, 118, 149, 151, 148, 237]. It should be mentioned that ansatz (1.8) was also applied in the earlier papers [200, 238], while later, in the 2000s, it was shown that the ansatz is applicable for reduction of some nonlinear PDEs when the functions $g_k = x^k$ contain rational (and even irrational) exponents k (see, e.g., examples in [84, 119]). Finally, a generalization of ansatz (1.8) for solving the multi-dimensional PDEs was suggested in [150] (see also examples in [119]).

Several cases of ansatz (1.7) with the functions $g_k \neq x^k$ were used for construction of new solutions to nonlinear evolution PDEs (see, e.g., [48, 52, 118, 237]). Paper [22, 200] should be quoted as one of the first in this direction.

In the 1990s, the method of linear invariant subspaces was developed [118, 237] and was extensively applied for constructing exact solutions of a wide range of nonlinear PDEs (almost all the results are summarized in monograph [119]). One may note that the method is reduced to finding exact solutions in the form (1.7) provided the functions $g_0(x), \dots, g_{m-1}(x)$ generate the basis of an invariant subspace for a given PDE.

Finally, the direct method [87], which is also called the Clarkson–Kruskal method, should be mentioned. The basic idea of the method is to reduce PDE (1.2) to ODE (a system of ODEs) using the ansatz

$$u = \varphi(t, x, \omega), \quad (1.9)$$

where the function $\omega(z)$ is a solution of some ODE, while the variable $z(t, x)$ should be additionally defined (e.g., it can be an invariant variable corresponding to a Lie symmetry). The function φ is usually assumed to be linear with respect to $\omega(z)$. It turns out that such ansatz leads to new exact solutions for some PDEs arising in real world applications [86, 87]. It was shown later that a consistency criteria can be formulated when the direct method and the nonclassical method [26] lead to the same solutions [13]. On the other hand, there are examples of the exact solutions found by the nonclassical method, which are not obtainable by the direct method [13, 193], hence the nonclassical method seems to be a more general tool for finding exact solution of nonlinear PDEs.

1.3 Lie symmetry of differential equations: historical review, definitions and properties

The Lie method was created by the famous Norwegian mathematician Sophus Lie. In the 1880s–1890s, he published several papers and books, which had an essential impact on development of pure and applied mathematics in the 20th century. It should be noted that his works were published in German and were not translated in English for a long time (to the best of our knowledge, some of them still exist in German only). The most important among Lie’s works from the applicability point of view are papers [166] and [167].

Research of S. Lie was inspired by the Galois group theory for solving algebraic equations. His aim was to create a similar theory for solving differential equations. In particular, he created *the theory of continuous groups, i.e., Lie groups* [170, 171, 172], which plays a key role in modern mathematics and its applications.

In the end of the 1950s, L.V. Ovsiannikov inspired by Birkhoff’s works on hydrodynamics [23], rewrote the basic notions of the Lie method using modern terminology and published the first monograph (after S. Lie) devoted to the

Lie method and its development [202]. His book accelerated a great interest to the subject during the 1960s–1980s. As a result, a huge number of papers and several books were published. The best known books (one may refer to them as classical) were published in that time [6, 27, 29, 114, 196, 203] (notably the Russian version of [114] was published in 1989). Many results derived during this period (and till the middle of the 1990s) are summarized in the collective monographs [7, 8, 136].

At the present time, the Lie method and related symmetry-based methods are still the most popular methods for construction of exact solutions of ODEs, PDEs (including multidimensional PDEs), integro-differential equations, difference equations and some others. There is a huge number of papers and several excellent books devoted to theoretical foundations of the Lie method and its application published in this millennium. In particular, the books [12, 24, 25, 135] are devoted to presentation of the Lie method and related symmetry-based methods. They are partly based on the relatively recent results and the modern terminology is used therein.

Taking into account that the Lie method is a well-established method with rigorous theoretical foundations, here we briefly present the basic notions and theorems only. The reader may find much more details in the monographs and textbooks cited above. Because this monograph is devoted mostly to two-dimensional PDEs, we restrict ourselves to the two-dimensional case from the very beginning. The relevant generalization on the multi-dimensional case can be done in a straightforward way (or found elsewhere).

Let us consider the k -order PDE

$$L\left(t, x, u, u_1, \dots, u_k\right) = 0, \quad k \geq 1, \quad (1.10)$$

where $u = u(t, x)$ is an unknown function, u means a totality of s -order derivatives of $u(t, x)$ ($s = 1, 2, \dots, k$) and L is a given smooth function of their arguments. On the other hand, we suppose that the transformations

$$\bar{t} = f^0(t, x, u, v), \quad \bar{x} = f^1(t, x, u, v), \quad \bar{u} = g(t, x, u, v) \quad (1.11)$$

(here $v = (v_1, \dots, v_r)$ are group parameters and f^0 , f^1 and g are the given smooth function) generate the r -parameter Lie group G^r . According to the known algorithm, the k -th order extension (prolongation) of G^r can be constructed (see for details, e.g., [24, Chapter 2]) acting in the prolonged space of the variables

$$t, \quad x, \quad u, \quad u_1, \quad \dots, \quad u_k. \quad (1.12)$$

In the prolonged space (1.12), each k -th order PDE can be considered as a manifold (surface), i.e., it is a geometrical object.

Definition 1.1 *The r -parameter Lie group G^r is called the invariance group (the group of point symmetries) of PDE (1.10), if its k -th order extension leaves invariant the surface defined by (1.10) in the prolonged space (1.12).*

It follows immediately from Definition 1.1 that an arbitrary solution of PDE (1.10) is mapped to another (or the same) solution of this equation under action of each transformation from G^r . So, the set of all solutions of PDE (1.10) is invariant under G^r .

It is well-known that there is a bijective correspondence between Lie groups and Lie algebras of the same dimensionality. Thus, having the invariance group G^r of PDE (1.10), one can easily construct the corresponding algebra of invariance and vice versa. In fact, the following statements take place.

Theorem 1.1 *Let us consider the continuous transformations, which correspond to the r -parameter group G^r (1.11). Then this group generates the r -dimensional Lie algebra AG^r with the basic operators*

$$X_j = \xi_j^0(t, x, u)\partial_t + \xi_j^1(t, x, u)\partial_x + \eta_j(t, x, u)\partial_u, \quad j = 1, \dots, r \quad (1.13)$$

where the functions ξ_j^0 , ξ_j^1 and η_j are determined as follows

$$\xi_j^0 = \left. \frac{\partial f^0(t, x, u)}{\partial v_j} \right|_{v=0}, \quad \xi_j^1(t, x, u) = \left. \frac{\partial f^1}{\partial v_j} \right|_{v=0}, \quad \eta_j(t, x, u) = \left. \frac{\partial g}{\partial v_j} \right|_{v=0}. \quad (1.14)$$

Theorem 1.2 *Let us consider the r -dimensional Lie algebra AG^r with the basic operators (1.13). Then this algebra generates the r -parameter Lie group G^r (1.11), where the functions $\bar{t} = f^0(x, u, v)$, $\bar{x} = f^1(x, u, v)$ and $\bar{u} = g(x, u, v)$ form a unique solution of the initial problem*

$$\begin{aligned} \frac{\partial \bar{t}}{\partial v_j} &= \xi_j^0(\bar{t}, \bar{x}, \bar{u}), \quad \frac{\partial \bar{x}}{\partial v_j} = \xi_j^1(\bar{t}, \bar{x}, \bar{u}), \quad \frac{\partial \bar{u}}{\partial v_j} = \eta_j(\bar{t}, \bar{x}, \bar{u}) \\ f^0(x, u, 0) &= t, \quad f^1(x, u, 0) = x, \quad g(x, u, 0) = u. \end{aligned} \quad (1.15)$$

Remark 1.1 *To guarantee a unique solution of the initial problem (1.15) (at least in a vicinity of the point $v = (v_1, \dots, v_r) = 0$), one should assume that the coefficient of the Lie operators (1.13) are sufficiently smooth.*

Theorems 1.1 and 1.2 lead to the following consequence: each invariance group of the given PDE is known provided the corresponding Lie algebra of invariance is found and vice versa. Because the operators of any Lie algebras create a linear vector space, usually it is more convenient to deal with Lie algebras if differential equations are under study. Let us present an example in order to illustrate the above definition and theorems.

Example 1.1 *Consider the nonlinear equation*

$$u_t = (uu_x)_x, \quad (1.16)$$

where $u(t, x)$ is an unknown function. One may say that Eq. (1.16) is a classical example among nonlinear parabolic equations because one is used in a wide range of applications. This equation was introduced by J. Boussinesq [32] and is often named after him (the name “porous diffusion equation” is also used).

Let us consider the set of transformations

$$\bar{t} = e^{2v_2}t + v_0, \quad \bar{x} = e^{v_2}x + v_1, \quad \bar{u} = u, \quad (1.17)$$

where v_0, v_1 and v_2 are arbitrary real parameters. One easily checks that it is the three-parameter Lie group G^3 . Assume that $u(t, x)$ is an arbitrary solution of the Boussinesq equation and examine the function $\bar{u}(\bar{t}, \bar{x})$. Taking into account (1.17), one obtains

$$\bar{u}_{\bar{t}} = e^{-2v_2}u_t, \quad \bar{u}_{\bar{x}\bar{x}} = e^{-2v_2}u_{xx}$$

so that $\bar{u}(\bar{t}, \bar{x}) = u(e^{2v_2}t + v_0, e^{v_2}x + v_1)$ is also the exact solution of Eq. (1.16). Thus, the Lie group G^3 is the invariance group of this equation.

Let us apply Theorem 1.1 for finding the corresponding Lie algebra. Substituting (1.17) into formulae (1.14), we easily calculate the coefficients of the basic operators:

$$\begin{aligned} \xi_0^0 &= \left. \frac{\partial(e^{2v_2}t + v_0)}{\partial v_0} \right|_{v=0} = 1, \quad \xi_0^1 = \left. \frac{\partial(e^{v_2}x + v_1)}{\partial v_0} \right|_{v=0} = 0, \\ \xi_1^0 &= \left. \frac{\partial(e^{2v_2}t + v_0)}{\partial v_1} \right|_{v=0} = 0, \quad \xi_1^1 = \left. \frac{\partial(e^{v_2}x + v_1)}{\partial v_1} \right|_{v=0} = 1, \\ \xi_1^0 &= \left. \frac{\partial(e^{2v_2}t + v_0)}{\partial v_2} \right|_{v=0} = 2t, \quad \xi_2^1 = \left. \frac{\partial(e^{v_2}x + v_1)}{\partial v_2} \right|_{v=0} = x, \\ \eta_j &= \left. \frac{\partial u}{\partial v_j} \right|_{v=0} = 0, \quad j = 0, 1, 2. \end{aligned} \quad (1.18)$$

Finally, substituting these coefficients into (1.13), we arrive at the three-dimensional Lie algebra with the basic operators

$$X_0 = \partial_t, \quad X_1 = \partial_x, \quad X_2 = 2t\partial_t + x\partial_x.$$

An operator of invariance (nowadays the terminology ‘‘Lie (point) symmetry’’ is widely used) has the form of a first-order quasilinear differential operator

$$X = \xi^0(t, x, u)\partial_t + \xi^1(t, x, u)\partial_x + \eta(t, x, u)\partial_u. \quad (1.19)$$

If (1.19) is a Lie symmetry of PDE (1.10) then the coefficients $\xi^0(t, x, u)$, $\xi^1(t, x, u)$ and $\eta(t, x, u)$ can be found using the definition (the invariance criteria).

Definition 1.2 Operator (1.19) is called Lie symmetry of PDE (1.10) if the following invariance criteria is satisfied:

$$X_k(L) \Big|_{\mathcal{M}} = 0, \quad (1.20)$$

where the differential operator X_k is the k -order prolongation of operator (1.19) and the manifold $\mathcal{M}$ is defined by the equation in question, i.e., it is a smooth manifold $L(t, x, u, u_1, \dots, u_k) = 0$ in the prolonged space of the variables (1.12)

The operator

$$X_k = X_{k-1} + \eta \cdot \partial_k u$$

(here $\partial_k u$ is the vector consisting a totality of the operators $\frac{\partial}{\partial u_k}$ with u_k belonging to the set u_k , while “ $\cdot$ ” means the scalar product) is the k th-order prolongation of the operator $X \equiv X_0$. Its coefficients are expressed via the functions ξ^0 , ξ^1 and η by the well-known formulae, which can be written in the form (see, e.g., [203])

$$\eta_k = D_{k-1} \eta - u D_k \xi. \quad (1.21)$$

In the case $k = 2$, one obtains

$$\eta_1 = (\zeta^0, \zeta^1), \quad \eta_2 = (\sigma^{00}, \sigma^{01}, \sigma^{11}), \quad D = (D_t, D_x),$$

$$D_t = \partial_t + u \partial_{u_t} + u_t \partial_{u_{tt}} + u_x \partial_{u_{tx}}, \quad D_x = \partial_x + u \partial_{u_x} + u_t \partial_{u_{xt}} + u_x \partial_{u_{xx}}.$$

Thus, operator X_2 , which is the second prolongation of the operator X , has the form

$$X_2 = X + \zeta^0 \partial_{u_t} + \zeta^1 \partial_{u_x} + \sigma^{00} \partial_{u_{tt}} + \sigma^{01} \partial_{u_{tx}} + \sigma^{11} \partial_{u_{xx}}, \quad (1.22)$$

where

$$\begin{aligned} \zeta^0 &= D_t \eta - u_t D_t \xi^0 - u_x D_t \xi^1, \\ \zeta^1 &= D_x \eta - u_t D_x \xi^0 - u_x D_x \xi^1, \\ \sigma^{00} &= D_t \zeta^0 - u_{tt} D_t \xi^0 - u_{tx} D_t \xi^1, \\ \sigma^{01} &= D_x \zeta^0 - u_{tt} D_x \xi^0 - u_{tx} D_x \xi^1, \\ \sigma^{11} &= D_x \zeta^1 - u_{xt} D_x \xi^0 - u_{xx} D_x \xi^1. \end{aligned} \quad (1.23)$$

It turns out that a set of all Lie symmetry operators of the given PDE always form a Lie algebra. Because a subset of this set may form another Lie algebra, the following notion is important.

Definition 1.3 *A Lie algebra of invariance of the given PDE is called the maximal algebra of invariance (MAI) if one contains each Lie algebra of invariance of this equation as a subalgebra. The Lie group, which corresponds to MAI, is called the maximal group of invariance (MGI).*

The problem of finding MAI for the given PDE is reduced to solving an overdetermined system of linear PDEs (although the equation in question can be nonlinear!). Nowadays it is a straightforward routine, which can be realized using computer algebra packages like **Maple**, **Mathematica**, **Reduce** etc (see, e.g., [85, 130, 248]).

However, the problem becomes nontrivial if PDE contains arbitrary functions as coefficients. In this case the *Lie symmetry classification (LSC) problem* (terminology “group classification problem” is also used) arises. Let us consider the class of PDEs

$$L \left(t, x, u, u_1, \dots, u_k, \theta \right) = 0, \quad (1.24)$$

where θ is an arbitrary smooth function (can be two or more functions), which depends on the variables t, x, u (dependence on derivatives of u can be as well). In the models describing real world processes, the function θ (with appropriate restrictions) has a clear physical (biological, chemical etc.) interpretation. Obviously, Lie symmetry of PDEs belonging to class (1.24) depends essentially on the form of θ .

Remark 1.2 *It may happen that MAI of PDEs involving θ as a parameter (not a function !) depends on the value of θ . The classical example is the diffusion equation with power-law diffusivity $d(u) = u^k$ (see Section 2.2).*

Two types of the LSC problem are usually distinguished.

1. To find all the correctly-specified forms of θ when PDE (1.24) admits a fixed Lie algebra.
2. To describe all possible Lie algebras (Lie groups), which can be admitted by PDEs from the given class depending on the form of θ .

The first type is very important for examination of mathematical models in theoretical physics because all classical equations (the heat and wave equations, the Laplace and Schrödinger equations, the Maxwell and Dirac systems, etc.) admit Lie symmetries, which reflect the fundamental laws of physics. If one is looking for some nonlinear analogs of the classical equations then the generalizations to be constructed should preserve the relevant Lie symmetry (the reader can find references to an extensive discussion on this topic in [64, 110, 114, 222]).

The second type is more popular among mathematicians who are dealing with PDEs arising in different branches of applied mathematics. In this case, the well-established class of PDEs is usually taken from literature and examined using the Lie method. Chapter 2 is devoted to such type of the LSC problems.

Now we turn to the operator X (1.19). Throughout the book we use the notation $X(u) = 0$ for the first-order PDE

$$\xi^0 u_t + \xi^1 u_x - \eta = 0, \quad (1.25)$$

which is generated by the Lie symmetry (1.19) and often is called the *invariance surface condition* or the characteristic equation of X . Rigorously speaking, one should write the invariance surface condition in the form (see, e.g., [24] for details)

$$X(u - u_0(t, x)) \Big|_{u=u_0(t, x)} = 0,$$

which leads exactly to the above first-order PDE for an arbitrary solution $u = u_0(t, x)$.

S. Lie established a remarkable property of Eq. (1.25), which allow us to apply Lie symmetries for the exact reduction of the given PDE to ODEs or algebraic equations (in the case of multidimensional PDEs such reduction is more complicated and usually leads to another PDE of a lower dimensionality). In fact, because (1.25) is the quasi-linear first-order PDE, its general solution can be always constructed in explicit (or implicit) form and one depends on a function of a single variable (two variables if the general solution is found in an implicit form). In the case of evolution PDEs (such equations are the main subject of this book), a typical form of the general solution of (1.25) is

$$u = g(t, x) + \varphi(\omega(t, x))G(t, x), \quad (1.26)$$

where φ is an arbitrary function of the variable $\omega(t, x)$, while $g(t, x)$ and $G(t, x)$ are the known functions. The above property guarantees that substitution (1.26) reduces the given PDE (1.10) to the equation

$$L_1(\omega, \varphi, \varphi_\omega^{(1)}, \varphi_\omega^{(2)}, \dots, \varphi_\omega^{(k_1)}) = 0, \quad (1.27)$$

which is ODE of the order $k_1 \leq k$ for the unknown function φ . Eq. (1.27) is called the *reduction equation*. Thus, each exact solution of the reduction equation (1.27) can be easily transformed into a particular solution of PDE (1.10) using formula (1.26).

Substitution (1.26) is called *ansatz* (Lie's ansatz) as it was proposed by S. Lie in his works. Taking into account that there are also ansätze, which are not related with Lie symmetry, the following definition is useful.

Definition 1.4 *A local substitution reducing a given PDE into another PDE (ODE or algebraic equation) with a smaller number of independent variables is called ansatz.*

If an ansatz is related with a Lie symmetry from MAI of the given equation then it is Lie's ansatz, otherwise the notion non-Lie ansatz will be used. Because each linear PDE admits infinite-dimensional MAI, which reflects the principle of linear superposition of solutions, an infinite number of Lie's ansätze can be constructed. On the other hand, there are several classical methods for solving linear PDEs, therefore we concentrate ourselves on nonlinear PDEs in what follows.

Definition 1.5 *An exact solution of PDE in question that can be found using a Lie symmetry operator (i.e., Lie's ansatz) and (possible) subsequent application of transformations from its invariance group is called invariant solution (group-invariant solution, similarity solution, Lie solution).*

Now we present the next example, which is a natural continuation of Example 1.1 and illustrate the above definitions.

Example 1.2 *Here we continue examination of Eq. (1.16). The Lie algebra of invariance derived in Example 1.1 is not maximal because MAI of Eq. (1.16) is the four-dimensional Lie algebra [203]*

$$AG^4 = \langle \partial_t, \partial_x, 2t\partial_t + x\partial_x, x\partial_x + 2u\partial_u \rangle. \quad (1.28)$$

Thus, taking any linear combination of the above operators, one can construct a Lie's ansatz. Notably, several Lie's ansätze were derived by ad hoc approaches (i.e., without application of the Lie method). For example, taking the linear combination $\partial_t + v\partial_x$ (here v is an arbitrary parameter, which usually means velocity) and using the invariance surface condition (1.25), the linear first-order PDE $u_t + vu_x = 0$ is obtained. So, the corresponding ansatz has the form

$$u = \varphi(\omega), \quad \omega = x - vt, \quad (1.29)$$

where $\varphi(\omega)$ is to be determined function. On the other hand, (1.29) is the well-known substitution for finding the plane wave solutions (particularly traveling fronts, solitary waves, kinks etc.) and is applicable for a wide range of PDEs.

Let us construct a nontrivial ansatz using the Lie symmetry $x\partial_x + 2u\partial_u$. In this case, the invariance surface condition takes the form $xu_x - 2u = 0$, hence one obtains the ansatz

$$u = x^2\varphi(\omega), \quad \omega = t. \quad (1.30)$$

Substituting ansätze (1.29) and (1.30) into Eq. (1.16), one derives two reduction equations

$$(\varphi\varphi_\omega)_\omega + v\varphi_\omega = 0, \quad \varphi_\omega = 4\varphi^2. \quad (1.31)$$

Obviously, both ODEs can be easily integrated, hence the corresponding particular solutions of the Boussinesq equation are obtained using formulae (1.29) and (1.30).

In contrast to the above cases, the problem of finding exact solutions for given nonlinear PDEs via application of Lie symmetries can be a nontrivial task. For example, if one takes the linear combination of the last two operators from AG^4 , i.e., $2t\partial_t + (1+v)x\partial_x + 2vu\partial_u$ then the corresponding ansatz and reduction equation are much more complicated (its analog will be examined in Section 4.2). As a result, invariant solutions of Eq. (1.16) cannot be constructed for an arbitrary parameter v . Notably, a solution corresponding to the parameter $v = -1$ firstly was derived in [32].

Because construction of exact solutions is the most important feature of the Lie method (we remind the reader that S. Lie initially was going to create a general theory of integrability of ODEs and PDEs), one needs to develop an algorithm how to derive (at least to describe) all invariant solutions of the equation in question.

The algorithm for constructing invariant solutions of nonlinear PDEs consists of the following main steps.

1. To find MAI of the equation in question using the invariance criteria (1.20).
2. To construct all inequivalent Lie's ansätze by solving the characteristic equations of the form (1.25) for the corresponding operators from MAI.
3. To construct the list of reduction equations using the ansätze derived.
4. To solve the reduction equations using the existing techniques and to construct exact solutions of the equation in question.
5. To construct formulae of multiplication of the solutions obtained using Definition 1.1 and group transformations from MGI.

It should be stressed that all the steps of the algorithm often cannot be realized because the fourth step essentially depends on the form of the reduction equations. So, some particular solutions of those can be constructed only. However, one may provide a qualitative analysis of nonintegrable reduction equations in order to establish properties of the solutions of PDE in question (see, e.g., [68]).

Other steps of the algorithm can be realized for a wide range of PDEs (including multidimensional ones) arising in real world applications. We briefly present some remarks to each step only. As it was already noted, MAI can be found using computer algebra packages (we assume that the given equation does not contain arbitrary functions as coefficients).

The second step is completed in a straightforward way provided MAI is of a low dimensionality (for example, the MAI dimensionality is $r = 2, 3$ or 4). In such cases, one takes a linear combination of the basic operators (with arbitrary coefficients, say, λ_i , $i = 1, \dots, r$) and obtains Eq. (1.25) involving r parameters. Solving such equation with a small number of arbitrary parameters usually is a simple task, so that all possible ansätze are derived. However, the task becomes a nontrivial problem if MAI has the high dimensionality (or one is infinite-dimensional). Nonlinear multidimensional PDE arising in real world applications usually are invariant under such algebras. For example, the well-known Schrödinger equation with cubic nonlinearity in 1D space (i.e., the two-dimensional PDE) is invariant under 5-dimensional MAI, while one in 3D space admits 12-dimensional MAI (see, e.g., [108, 109, 114]). In the case of MAI of a high dimensionality, the additional problem occurs, which was formulated by Ovsiannikov in the 1960s [202, 203]. One needs to construct systems of inequivalent (nonconjugated) subalgebras of MAI of the

dimensionalities $1, \dots, r - 1$. Each such system is called the optimal system of n -dimensional subalgebras. It turns out that this problem, which is a pure algebraic one, can be completely solved for all possible Lie algebras of the low dimensionalities [204, 205, 206] but the problem is still unsolved in the general case. To the best of our knowledge, the problem is not solved for six- and higher-dimensional Lie algebras at the present time (see also the extensive discussion in [163]). In the case of two-dimensional PDEs, the problem simplifies because one needs to construct only the optimal system of one-dimensional subalgebras for the given Lie algebra.

The third step of the algorithm consists of simple calculations provided all inequivalent ansätze are derived.

The last step of the algorithm is not important from the rigorous mathematical point of view. In fact, one can claim that all solutions of the given PDE are equivalent (although they may have very different properties!) if they can be mapped to a fixed invariant solution by a transformation from MGI. Thus, it is enough to construct the so-called optimal system of invariant solutions corresponding to subalgebras belonging the given optimal system (see, e.g., [196]). However, the fifth step is important from applicability point of view. Let us take an exact solution, which is trivial and without any physical (biological) interpretation. The solution can be multiplied to a set of solutions by the transformations belonging to MGI. As a result, some of the solutions obtained may admit an interpretation of the process described by the equation in question. For example, the constant solution of the classical heat equation $u_t = u_{xx}$ can be mapped to its fundamental solution $u = \frac{1}{2\sqrt{t}} \exp(-\frac{x^2}{4t})$ by the invariance transformations (and taking a limiting case) of the equation. Notably, the relevant formula of the multiplication of exact solutions of the heat equation was derived by P. Appell [9] without knowledge of the Lie method. On the other hand, this formula was generalized on some nonlinear PDEs and nonlinear systems of PDEs using the Lie method (see examples in [44, 50, 108, 109, 110]).

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