AN INTRODUCTION TO GAUGE THEORIES

Nicola Cabibbo
Luciano Maiani
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Preface

Introduction to Gauge Theories completes the series of three volumes based on lecture courses in relativistic quantum mechanics, electroweak interactions and gauge theories, taught by the authors to first-year master's degree students in physics and astronomy, and astrophysics (*Laurea Magistrale*) of the University of Rome, "La Sapienza", over a period of several decades.

The principal objective of this volume is to introduce the basic concepts of renormalisation in quantum field theory and the fundamentals of modern gauge theories. Even though it is connected to the previous volumes, the book can be read independently; it assumes only a general familiarity with special relativity, second quantisation and the phenomenology of electroweak interactions.

The basic tool is the Feynman path integral, introduced in the early chapters and systematically employed in what follows. The exposition follows a pedagogic course, which begins with the simple case of the quantum mechanical transition amplitude to lead to the generating functional of the Green's functions of field theory. The same pedagogic approach is used in the chapter in which equations of motion, symmetries and the Ward identity are discussed. The analysis of the path integral formalism is completed by a discussion of anticommuting variables and the quantisation of fermion and electromagnetic fields.

The perturbative expansion of the generating functional of the Green's functions, and the Lehman, Symanzyk and Zimmermann reduction formulae—which allow the latter to be related to the scattering matrix elements—are illustrated, first in the simple case of a scalar field theory with $\lambda\phi^4$ interactions and subsequently with quantum electrodynamics (QED).

Renormalisation in QED is treated in the central part of the book. The appearance of ultraviolet divergences, from which the necessity of introducing a regularisation procedure follows, is illustrated by taking as examples corrections of order α to the photon and electron propagators and to the interaction vertex. The fundamental Ward identity of QED is also proven to the same perturbative order.

The discussion of QED to second order is supplemented by the detailed illustration of several important applications, which include analysis of the infrared divergence, detailed calculations of the Lamb shift and the vacuum polarisation tensor, and Schwinger's calculation of the correction to the anomalous magnetic moment of the electron, extended in a subsequent chapter to

the full Electroweak corrections in the Standard Theory. Introduction to the renormalisation group completes the part dedicated to QED, with discussion of the Gell–Mann and Low equation, and the Landau pole.

In the following part of the book, quantisation of non-Abelian gauge theories is analysed, and the evolution of the coupling constant with the energy and momentum scale. Asymptotic freedom in quantum chromodynamics, the fundamental theory of strong interactions, is demonstrated to second perturbative order through explicit calculation of the β function. The evolution of the coupling constant of the fundamental interactions to high and very high energies is discussed as an introduction to the hypothesis of grand unification.

The recent discovery of the Higgs boson has reopened discussion of the naturalness, or non-naturalness, of scalar fields in field theory, also in connection with the possible presence of new physics at energies higher than the critical energy of the Standard Theory. The final chapters are dedicated to the study of effects of scalar fields in the Standard Theory, analysis of limits on the mass of the Higgs boson, the calculation of quantum corrections to the effective potential of scalar fields and the problem that naturalness of elementary scalars poses for the structure of ultraviolet divergences of the potential.

A large part of this work is due to Nicola Cabibbo with whom we have been linked in a collaboration that lasted several decades. Nicola's contributions, several of which are on subjects not usually treated in introductory texts on gauge theories, are unmistakable for their originality and clarity.

In conclusion, we would like to express particular gratitude to Guido Altarelli, Riccardo Barbieri, John Iliopoulos, Gino Isidori, Giorgio Parisi, Antonio Polosa, Veronica Riquer, Massimo Testa, and to many other colleagues for numerous conversations and comments on the subject, and also express our affectionate thanks to Paola Cabibbo, for the patience with which she has followed the slow process of writing. Finally, we would like to acknowledge the fact that this book owes much to the comments we have received from our students, over the course of the years.

Luciano Maiani and Omar Benhar

Rome, August 2016

The Authors

Luciano Maiani is emeritus professor of theoretical physics at the University of Rome, “La Sapienza”, and author of more than two hundred scientific publications on the theoretical physics of elementary particles. Together with S. Glashow and J. Iliopoulos, Maiani made the prediction of a new family of particles, those with “charm”, which form an essential part of the unified theory of the weak and electromagnetic forces. He has been president of the Italian Institute for Nuclear Physics (INFN), director-general of CERN in Geneva and president of the Italian National Council for Research (CNR). He has promoted the development of the Virgo Observatory for gravitational wave detection, the neutrino beam from CERN to Gran Sasso and at CERN has directed the crucial phases of the construction of the Large Hadron Collider (LHC). He has taught and worked in numerous foreign institutes. He was head of the theoretical physics department at the University of Rome, “La Sapienza”, from 1976 to 1984 and held the chair of theoretical physics from 1984 to 2011. He is a member of the Italian Lincean Academy and a fellow of the American Physical Society. For his scientific work, he has been awarded the J. J. Sakurai Prize, the Enrico Fermi Prize, the Dirac Medal, the High Energy and Particle Physics Prize of EPS and the Bruno Pontecorvo Prize.

Omar Benhar is an INFN research director and teaches gauge theories at the University of Rome, “La Sapienza”. He has worked extensively in the United States as a visiting professor at the University of Illinois and Old Dominion University, and was an associate scientist at the Thomas Jefferson National Accelerator Facility. Since 2013, he has served as an adjunct professor at the Centre for Neutrino Physics of Virginia Polytechnic Institute and State University. He is the author of more than one hundred scientific papers on the theory of many-particle systems, the structure of compact stars and electroweak interactions of nuclei.

Nicola Cabibbo (1935–2010) was professor of Theoretical Physics and Elementary Particle Physics at the Rome Universities *La Sapienza* and *Tor Vergata*, and held research and teaching positions in prestigious institutions such as Harvard University, the Institute for Advanced Studies, Princeton, CERN, Geneva, University of California at Berkeley and Université Paris VI. In 1962, Cabibbo discovered the phenomenon of quark mixing, described by a new nat-

ural constant, the *Cabibbo angle*, measured with great accuracy in semileptonic weak decays of hadrons. According to a recent analysis, Cabibbo's paper on quark mixing was the most influential article published in the journals of the American Physical Society during 1893–2003. In the 1980s, Cabibbo provided important momentum to the applications of numerical techniques to theoretical physics, notably the gauge theories of strong interactions, promoting and leading the development of the family of APE (Array Processor Experiment) supercomputers. He served as a member of a number of learned societies: Accademia Nazionale dei Lincei and Accademia delle Scienze di Torino, in Italy, National Academy of Science and American Association for Art and Sciences, in the United States, and Accademia Pontificia delle Scienze, which he chaired from 1993. An internationally reputed science manager, Cabibbo was president of Istituto Nazionale di Fisica Nucleare (INFN) and of Ente Nazionale per le Nuove Tecnologie per Energia e Ambiente (ENEA). He was the recipient of the J.J. Sakurai Prize (APS), the Medaglia Matteucci (Accademia Nazionale dei XL), the Dirac Medal (ICTP Trieste), and the Benjamin Franklin Medal.

INTRODUCTION

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1.1 QUANTUM ELECTRODYNAMICS

In this course we will largely focus on quantum electrodynamics, the field theory which describes the interaction of charged particles with an electric field. Quantum electrodynamics, or QED, is an incomplete theory, given that all the elementary particles are also subject to the weak interaction and, in the case of quarks, also to the strong interaction.

A more complete theory, which takes account of both electromagnetic and weak and strong interactions, is provided by the so-called Standard Theory (or Standard Model). The Standard Theory is also incomplete, given that it does not take account of gravitational interactions.¹

Despite its limitations, QED is very interesting for several reasons:

- QED has a wide range of interesting physical applications, from the interactions of photons and electrons to the fine structure of matter at the atomic level.
- QED was the first field theory to have been studied in detail, mainly using the method of Feynman diagrams, to address fundamental issues such as the occurrence of infrared and ultraviolet divergences and renormalisation.
- QED is a gauge theory, and is therefore the prototype of the Standard Theory. A study of QED is a valuable introduction to the study of the Standard Theory.

¹Although at the level of classical (non-quantum) physics, gravitational interactions are described very successfully by Einstein's general theory of relativity, and in many cases even Newton's theory can be satisfactorily adequate, there still does not yet exist a universally accepted quantised form of gravity. The most widely held opinion is that it will be necessary to progress from a field theory to a string theory, an argument beyond the scope of this course.

FERMIONS			matter constituents		
Leptons spin = 1/2			Quarks spin = 1/2		
Flavor	Mass [Gev]	Electric charge	Flavor	Approx. Mass [GeV]	Electric charge
ν_e <i>electron neutrino</i>	$<1 \times 10^{-8}$	0	u up	0.003	2/3
e electron	<0.000511	-1	d down	0.006	-1/3
ν_μ <i>muon neutrino</i>	<0.0002	0	C charm	1.3	2/3
μ muon	0.106	-1	S strange	0.1	-1/3
ν_τ <i>tau neutrino</i>	<0.02	0	t top	175	2/3
τ tau	1.7771	-1	b bottom	4,3	-1/3

Figure 1.1 Fundamental fermions.

- Among the predictions of the Standard Theory, some of those related to QED are confirmed with the highest precision.

The most precise test of QED is currently given by the experimental value of the anomalous magnetic moment of the electron. The Dirac equation assigns to the electron a magnetic moment equal to one Bohr magneton², $\frac{e}{2m}$, but this result must be corrected because of the interaction with the radiation field by a factor $(1 + a_e)$, where a_e is the *magnetic anomaly*, which can be expressed as a power series in α , the fine structure constant:

$$a_e = \frac{\alpha}{2\pi} + \cdots .$$

The experimental values for the electron and positron are known with

²Units and other conventions are discussed at the end of this chapter.

errors of the order of 0.004 parts per million:

$$\begin{aligned} a_{e-}^{\text{Exp}} &= (1159652.1884 \pm 0.0043) \times 10^{-9} , \\ a_{e+}^{\text{Exp}} &= (1159652.1879 \pm 0.0043) \times 10^{-9} , \end{aligned} \quad (1.1)$$

to be compared with the theoretical prediction:

$$a_{e-}^{\text{Th}} = (1159652.1535 \pm 0.0240) \times 10^{-9} \quad (0.02 \text{ parts per million}). \quad (1.2)$$

The error quoted for the theoretical prediction is mainly due to the uncertainty of the value of α , obtained from a measurement of the quantum Hall effect. The theoretical prediction is based on calculations of a_e up to terms in α^4 .

In this course we will discuss quantum electrodynamics using the method of Feynman path integrals, which has been shown to be definitely superior to the more traditional method of canonical quantisation [1] to handle quantum theories characterised by gauge symmetries. The concepts developed here provide a useful foundation to discuss the Standard Theory [2].

We recall that electromagnetism is characterised by the invariance of all the observable quantities, for example the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, with respect to transformation of the vector potentials:

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu f , \quad (1.3)$$

where f is an arbitrary function. We can consider this transformation as due to a transformation operator U_f , and write

$$U_f A_\mu U_f^{-1} = A_\mu + \partial_\mu f . \quad (1.4)$$

If we carry out two successive transformations, U_f , U_g , we will have

$$U_g U_f A_\mu U_f^{-1} U_g^{-1} = A_\mu + \partial_\mu f + \partial_\mu g , \quad (1.5)$$

and we can easily verify that the set of U_f form a commutative (Abelian) group:

$$U_g U_f = U_f U_g = U_{g+f} . \quad (1.6)$$

In the case of electromagnetism, we are therefore confronted with a particularly simple type of gauge invariance, while in the Standard Theory we must deal with non-commutative (non-Abelian) gauge symmetries. Even if the Feynman path integral method is not indispensable for the quantum description of electromagnetism, it is preferred in the case of the Standard Theory.

1.2 UNITS AND OTHER CONVENTIONS.

Heisenberg and Schrödinger representations. In what follows, we will principally use the Heisenberg representation, where the operators have a dependence on time:

$$O(t) = e^{itH} O e^{-itH} , \quad O = O(0) . \quad (1.7)$$

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We will note in passing when it will be preferable to use the Schrödinger representation. In each case, operators and states which will not be shown with any explicit dependence on time, for example $O|m\rangle$, are meant to be operators and fixed states, equivalent to $O(t=0)|m\rangle$ in the Heisenberg representation and to $O|m, t=0\rangle$ in the Schrödinger representation.

Units. We will use the system of units in which $\hbar = 1$ and $c = 1$.

Other conventions. We will follow the conventions used in [1, 3] for 4-vectors and the Dirac matrices. In particular, the scalar product between two 4-vectors $p = \{p^0, \mathbf{p}\}$ and $q = \{q^0, \mathbf{q}\}$ will be denoted as $pq \equiv p^\mu q_\mu = p^0 q^0 - \mathbf{p} \cdot \mathbf{q}$.

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