DIFFERENTIAL EQUATIONS

**A Problem Solving Approach
Based on MATLAB®**

P. Mohana Shankar

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A Problem Solving Approach Based on MATLAB[®]

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*Dedicated to my parents, Padmanabharao and
Kanakabai, who were school teachers*

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Preface

This book grew out of instructional material and notes the author created while teaching courses in Linear Algebra and Differential Equations to undergraduate engineering students. Starting from simple demos prepared for instructing students, all aspects associated with the first order differential equations, linear second and higher order differential equations with constant coefficients and coupled first order differential equations were incorporated into several modules for obtaining solutions to any exercise. Taking the pulse of the students to determine what they felt was essential for their understanding, these modules were constantly updated and put in to a final form where they display solutions to differential equations with a simple input in textual form. The solutions are displayed as figures such that they provide theoretical aspects, explanations, step-by-step generation of solutions, solutions using multiple approaches, including verification through numerical techniques. The examples in the book capture this holistic approach to problem solving. MATLAB® (version 2016a) with its symbolic toolbox provided the appropriate software for the creation of the solutions in this format.

The book provides theoretical background containing a large number of examples, far exceeding what is seen in typical textbooks and a substantial number of exercises. The goal of the book has been the creation of materials that provide the theory, ample examples to cover the diverse theoretical concepts in each chapter, and multiple ways of solving problems providing the pedagogy in terms of verification of solutions, etc. The solutions manual is available separately.

This book has been a family project with the support and encouragement from my wife Raja and daughter Raji. Their assistance in the preparation of materials was crucial to the timely completion of the project. I want to thank my editor Mr. Vijay Primlani for his support during this endeavor. The support provided by the staff at CRC press is also greatly appreciated.

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CHAPTER 1

Introduction

In the sciences, engineering, social sciences, business and other areas that impact our lives, changes take place continuously. This dynamic behavior can be easily demonstrated, studied and critically analyzed through modeling carried out by examining how changes take place in time. This time dependent or temporal behavior leads us to the notion of derivatives and differential equations. Therefore, a study of differential equations, their solutions, and interpretation of the solutions is essential for the proper understanding of systems operating in all walks of life.

Consider a phenomenon where the observable quantity is identified by y . If changes are taking place in time, it is obvious that y will be a function of time t , expressed as $y(t)$. If we take the first order derivative of y (differentiation once with respect to the independent variable t) and equate it to another function $f(y,t)$, we obtain a first order differential equation expressed as

$$\frac{dy}{dt} = f(y, t). \quad (1.1)$$

It should be noted that $f(y,t)$ may be a constant, a function of time, a function of y (implicit dependence on t) or a function of y and t . For example, we may have

$$\frac{dy}{dt} = f(y, t) = y + y^2 t + t^2 + 2. \quad (1.2)$$

If $f(y,t)$ only depends on y , the first order differential equation is an **autonomous** one as in the case of

$$\frac{dy}{dt} = f(y, t) = h(y). \quad (1.3)$$

In eqn. (1.3) $h(y)$ is a linear or nonlinear function of y only.

It is possible to take the derivative of eqn. (1.1) once more. In this case, we get a **second** order differential equation. We may also take higher order derivatives. The order of the differential equation is determined by the highest order derivative existing in the equation. In addition to differential equations in a single variable of the first or higher orders, it is also possible that there may be two or more observables such as $y_1(t)$, $y_2(t)$,..., in which case, we may have interconnected or coupled differential equations.

In this book, we study all these differential equations starting from a single first order differential equation, followed by second order and higher order differential equations and culminating in coupled first order differential equations. If the differential equations only contain terms in y , $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, ..., or $\frac{d^n y}{dt^n}$, the differential equations are **linear**. An example of a **4th order linear differential** equation is

$$a(t)\frac{d^4 y}{dt^4} + b(t)\frac{d^2 y}{dt^2} + c(t)y = h(t). \quad (1.4)$$

If $a(t)$, $b(t)$ and $c(t)$ in eqn. (1.4) are constants (no dependence on t), the differential equation is identified as a **linear differential equation with constant coefficients**. An example of a second order linear differential equation with constant coefficients (A , B , and, C) is

$$A\frac{d^2 y}{dt^2} + B\frac{dy}{dt} + Cy = h(t). \quad (1.5)$$

Additionally, if $h(t)=0$, eqn. (1.5) will be a **homogeneous** differential equation while a **non-homogeneous** differential equation will have $h(t)\neq 0$. Note that $h(t)$ may be number (not equal to zero) in a non-homogeneous differential equation.

Chapter 2 is devoted to the study of first order differential equations with special emphasis on autonomous systems to develop an appreciation and understanding of the equilibrium conditions of associated systems. The chapter starts with the method of integrating factors and the method of separable functions for obtaining the solution of first order differential equations. A description of D-fields is provided to offer a pictorial description of the behavior of the system. All aspects necessary to understand an autonomous system are presented and well supported by an extensive set of examples. The examples are annotated to provide insights into the form of the solution, type of equilibrium as well as verification of the solution through additional approaches including use of `dsolve(.)` in MATLAB®, along with plots if and when necessary.

The second order differential equations are studied in **Chapter 3**. The differential equations are limited to linear ones with constant coefficients. Starting with the homogeneous differential equations, the concept of the characteristic equation or polynomial is introduced and the evolution of the solution from the roots of the characteristic equation is described. Solutions are also obtained using Laplace transforms and the state of the system is examined using the phase portraits to ensure that the analytical solution and their properties match the conclusions that can be drawn from phase portraits. An additional approach for obtaining the solution based on converting the second order equation into a pair of coupled first order differential equations is also offered as yet another level of verification of the solutions along with numerical solution obtained using Runge-Kutta methods. Finally, examples are provided which encompass every aspect of the solution with necessary equations, annotations, explanations and verification. Non-homogeneous differential equations are studied next in **Chapter 3** by offering two methods for obtaining the particular solutions, namely the method of undetermined coefficients and the method of variation of parameters. The particular solutions obtained using both methods are compared offering a comprehensive view of the solutions to the non-homogeneous differential

equations. The Laplace transform based approach and Runge-Kutta method are used to provide verification. Numerous examples are provided to cover diverse forcing functions and annotations in figures that clearly illustrate steps in the generation of the particular solutions, e.g., the need to scale the initial guess by t or t^2 as the case may be. The examples are fully annotated including steps taken to evaluate unknown constants when the initial conditions are given along with the background theoretical aspects.

Higher order differential equations are studied in [Chapter 4](#). The solution to the homogeneous differential equation is obtained using the roots of the characteristic polynomial and issues with the diversity of relationships among the multiple roots are discussed. Based on this, the characteristic polynomial is used only for differential equations up to the 4th order. The Laplace transform method is used for all orders. The particular solution is obtained using the method of variation of parameters (up to the 4th order). The Runge-Kutta method is used as a means to verify the results. Once again, the examples are annotated with theoretical aspects, relationships among the roots (if they exist), formulation of the solution, determination of the unknown constants (initial conditions given), and, plots of the theoretical and numerical solutions.

[Chapter 5](#) is devoted to coupled first order systems. A pair of coupled equations is studied first. The use of eigenvectors and eigenvalues in obtaining the solution to the homogeneous set is presented. The notion of defective matrices and the need for generalized eigenvectors is described. Another method based on the conversion of a pair of first order equations into a single second order equation is also presented along with the method based on Laplace transforms, as additional means for verification. The Runge-Kutta method is used as an extra step of verification. Phase portraits are used to justify conclusions about the stability of the system based on the solution obtained analytically. The concept of the fundamental matrix is invoked to obtain the particular solution when forcing functions are present. The fundamental matrix is used to obtain solutions when multiple coupled differential equations are present. Taking note of the existence of defective matrices, the analysis is limited to coefficient matrices of size $[4 \times 4]$ for the use of eigenvalues and eigenvectors. For larger matrices, only the Laplace transform based method (which is used throughout regardless of the size of the coefficient matrix) and the Runge-Kutta method are used for verification. The examples are annotated by displaying eigenvalues and eigenvectors, identifying defective matrices (algebraic and geometric multiplicities), degenerate eigenvalues, generalized eigenvectors, etc.

Appendices provide in depth coverage of topics such as numerical methods for solving differential equations (Euler's and Runge-Kutta methods), theory of Laplace transforms and applications to differential equations and phase plane analysis. In all cases, theoretical aspects are enunciated with appropriate examples to demonstrate the relevance to the issues presented in the main body of the book. The final appendix ([Appendix D](#)) is devoted to concepts of linear algebra that are absolutely essential for gaining a better understanding of the methods employed to find solutions to differential equations. These include properties of matrices, solution of a set of equations using row reduced echelon forms, Cramer's rule if applicable, matrix inversion if permitted and most importantly, the topic of eigenvalues and eigenvectors of square matrices. An expanded view of defective matrices and generalized eigenvectors is presented to

illustrate how they play an important role in obtaining solutions to a set of first order coupled differential equations.

A common thread throughout this book is the use of MATLAB (version 2016a) for the creation of the displays associated with every example. In every chapter, MATLAB scripts for solving example problems, creating phase plots, obtaining Laplace transforms, etc. are given for the benefit of the reader. Through the use of LaTeX conversion and extensive use of symbolic toolbox, the displays of the examples and solutions provide equations, explanations, sketches, plots, etc. reproducing the class room work by an instructor. MATLAB is used not merely as a means to verify the solution. It is also used to create a step-by-step solution matching the theory in every respect and thus providing a pedagogical element which is a unique feature of this book. Even though it is not possible to assure the absolute accuracies, every effort has been made to ensure that the MATLAB scripts (written in version 2016a) and the results provided are correct.

As indicated in the preface, the book is the culmination of the author's participation in teaching courses in Linear Algebra and Differential Equations to engineering students. The books used during that period of instruction and a number of journal publications by the author that came out of the preliminary work are listed below.

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A number of excellent books and monographs devoted to the topic of differential equations and linear algebra might provide supplementary information beyond what is covered in this book. Additionally, several books are also available that describe the use of MATLAB® in differential equations and linear algebra. A good source of information on the application of MATLAB is available within the help window of MATLAB itself. A partial list of some of these resources is provided below for additional reading.

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