The elastic local buckling behaviour and strength of the simply supported I-beam utilising the energy equilibrium method

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ABSTRACT: One of the phenomena influencing the capacity of the structure, or the ultimate state, is local buckling of the frame's members. This article presents the analyses of the most common setup, the simply supported beam with an I-section, and proposes the equations to estimate the local buckling strength of such beam. The derivation of the solution is based upon the principles of the theory of elastic stability, the shell stability, and the equilibrium of the work done by external forces and the plate's strain energy. Solutions are proposed for the segments of the beam, depending on the geometrical properties of those, and are later connected into equations considering the coupled interactions between the web and flanges. Finally, the equation allowing for the precise estimation of the strength of the whole beam is proposed.

1 INTRODUCTION

1.1 Previous research

Local buckling is one of the common forms of the stability loss occurring within the section, without the lateral displacement or twist of the whole beam. Local buckling has a significant influence on the behaviour of the whole structure, drastically reducing the frame's capacity or leading to global loss of stability i.e. the collapse of the structure (Fujak & Kimura 2019).

Full understanding of the local buckling phenomenon allows the engineer to correctly approximate the ultimate state of the structure, securing the linear behaviour in the elastic stage and nonlinear behaviour influenced only by the material properties in the plastic stage. Moreover, it allows for the correct classification of the section and determination if the members can stand the yielding load (sections class 1-3, see Eurocode 1993-1-1 (2005)), or are endangered by the local stability loss in the elastic work range (class 4). Additionally, the correct method of the analyses may be determined by the members elastic buckling strength. α_{cr} is the value of the elastic critical buckling load for the global instability mode based on initial elastic stiffness, F_{cr} , divided by the design loading on the structure, F_{Ed} . Eurocode 1993-1-1 (2005) allows for the use of the first-order analysis only when α_{cr} is higher than 10 for the elastic analyses and 15 for plastic analyses. Therefore, understanding of the elastic buckling strength of the members is crucial to correctly calculate the capacities of the structure.

Previous research on local buckling capacity is based upon the Euler-Bernoulli beam theory and Kirchhoff-Love plate theory. Timoshenko & Gere (1961) proposed the solutions for plates with simple boundary conditions and certain aspect ratios. With limited access to solving tools however, not all the solutions were given, and the effects of the flange-web interactions were not analysed. Roberts & Jhita (1983) analysed the problem of the elastic buckling of the I-beam focusing mostly on the strength of the flange. Bradford & Hancock (1984) described the method to consider the uneven stress distribution in wide-flange sections in the post-buckling stage.

Most of the recent research work is related to the problem of elastoplastic buckling. Ikarashi (2003) proposed the method of estimating the elastic buckling capacity of the whole beam

based on the buckling capacity of the web. However, this research was conducted to explain the plastic behaviour of the beam, so the precision of results in the elastic stage is moderate. Later works (Ikarashi et al. 2011, Kimura et al. 2019) proposed the indexes for the beam classification with respect to both length of the beam and loading applied to the element. These researches, however, focus on the capacity of the elements in the plastic stage.

To overcome the previously mentioned limitations research on the elastic buckling capacity of the simply supported beams is necessary. The aim of this study is:

- To investigate the strength of the plates (web, flanges) – segments building up the simply supported beam depending on their aspect ratios, with the effects of interaction between the segments considered.
- To propose simplified equations allowing for the estimation of local buckling capacity of the simply-supported beam.

1.2 Research proposal

The research presented analyses the elastic buckling strength of simply supported I-beams. Two types of commonly used loading are used – point load (hereinafter PL) and uniformly distributed load (UDL). The beam is as presented in Figure 1. Figure 1a presents the beam loaded with PL and the resultant stresses in the segments of the beam – the web and the flange. In the same manner, Figure 1b presents the loading and stress distribution in the beam loaded with UDL. Segments presented have simplified boundary conditions (BC) – either pinned or fixed. In reality, BC of the separate segments are semi-stiff, as the stiffness of the web influences the rotation of the flange and vice-versa. Note that the beams are ended with stiff diaphragm, which control node has certain degrees of freedom blocked ($\delta_x, \delta_y, \delta_z, \varphi_x, \varphi_z$ in case of the left-hand side end and $\delta_y, \delta_z, \varphi_x, \varphi_z$ in case of the right-hand side end). The stresses and deformations of the beam/plates are symmetric about the vertical axis of the beam. x, y, z are axes, L stands for the half of the length of the beam/plate, P_i for the PL in the i direction, q_i for the UDL in the i direction, M_i for the bending moment around the i axis, δ_i for the deflection in the i direction, φ_i for the rotation around the i axis, $\sigma_i(j, k)$ for the axial

Figure 1. Simply supported beam – analysed BCs, loads and stresses. a) Simply supported beam loaded with PL, b) Simply supported beam loaded with UDL.

Figure 2. Beam's section and segments' BC.

stress in the i direction as a function of j and k coordinates and $\tau_{ij}(k, l)$ for the shear stress in the i, j plane).

In order to analyse the section, it is crucial to understand the local behaviour of its segments. Therefore, the beam is first separated into segments, web and flanges, with simplified BC (Figure 2). The analyses of the detailed BC of web/flange influence is addressed in section 4.

As the flange is symmetrical about the vertical axis, it can be assumed to be built of two identical half-flanges. Note that B stands for the beam width, H for the height, b_f for the half-flange width, h_w for the web height, t_f for the flange thickness and t_w for the web thickness.

The proposed research is based on the energy equilibrium between the work done by external forces and the strain energy of the plate, see (1-4) (Timoshenko & Gere 1961).

$$\Delta T = \Delta U \quad (1)$$

$$\Delta T = -\frac{1}{2} \iint \left[N_x \left(\frac{\partial w}{\partial x} \right)^2 + N_y \left(\frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy \quad (2)$$

$$\Delta U = \frac{D}{2} \iint \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy \quad (3)$$

$$D = Et^3 / [12(1-\nu^2)] \quad (4)$$

where ΔT stands for the work done by external forces, ΔU for the plate's strain energy, N_x for the plate force in x direction, N_y for the plate force in y direction, N_{xy} for the shear force, $w(x, y)$ for the deformation of the plate in local z direction, ν for the Poisson's ratio of the material, E for the Young modulus of the material and t for the thickness of the analysed plate.

2 THE ELASTIC LOCAL BUCKLING OF SIMPLY SUPPORTED BEAM'S FLANGE

The models of flange used were of the following parameters: L of 600, 1000, 1500 and 2000 mm, b_f of 25, 50 and 100 mm, t_f of 2 and 7 mm for b_f of 25 and 100 mm, t_f of 2, 2.5, 3, 3.5, 4, 5, 6, 7, 8 and 10 mm for b_f of 50 mm.

The solution for the fixed flange is based on the shape of buckling mode as given in (5), $\delta_z = \varphi_x = 0$ along the long edge (Figure 1). The solution depends on values of the m and n parameters, related to the plate's aspect ratio L/b_f and loading conditions – either PL or UDL. The shape of the eigenform is checked against the finite element analyses (hereinafter: FEA) output.

$$w(x, y) = (x/L)^m \sin(n\pi x/L) \sin(\pi x/L)^2 (y/b_f)^2 \quad (5)$$

$$\sigma_{cr,f,fix} = Et_f^2 [\alpha_1 L^4 + \alpha_2 L^2 b_f^2 (1 - 1.5\nu) + \alpha_3 b_f^4] / [100\,000 L^2 b_f^4 (1 - \nu^2)] \quad (6)$$

$$\alpha_{1,PL} = 50\,000 (b_f/L)^2 \quad \alpha_{1,UDL} = 30\,000 (b_f/L)^2 \quad (7a, 7b)$$

$$\alpha_{2,PL} = 65\,000 + 500\,000 (b_f/L) \quad \alpha_{2,UDL} = 65\,000 + 500\,000 (b_f/L) \quad (8a, 8b)$$

$$\alpha_{3,PL} = 600\,000 + 17\,500 (L/b_f)^2 \quad \alpha_{3,UDL} = 700\,000 + 25\,000 (L/b_f)^2 \quad (9a, 9b)$$

For all the solutions, despite of the plate's aspect ratio, the final solution takes form of (6), where $\sigma_{cr,f,fix}$ stands for the elastic buckling stress of the plate. The value of parameters α_1 , α_2 and α_3 depends on the parameters described above. The values received from closed form (CF) solutions of (6) are presented in Figure 3 with markers. Plots of proposed (7-9) are given as well using dotted and dashed lines.

In (6), parameters α_1 and α_2 are used as the multipliers of the L^4 and $L^2 b_f^2$, respectively. As the length of the plate is used, parameters are convergent to the certain value. In case of α_1 , the parameter can be treated as constant when L/b_f is over 30, in case of α_2 (less dependent on L), the parameter stabilises with L/b_f over 50. This is not the case for α_3 , which is a multiplier for the b_f^4 . For the estimation of the equation the least squares method was used, with the additional restriction that α_1 , α_2 and α_3 must always be positive.

The identical procedure is implemented for the pinned flange model. Varying assumptions and results are given below in (10-13). Note that $\sigma_{cr,f,pin}$ stands for the elastic buckling stress of the plate. Results from FEA models of plates can be replicate by proposed (12 and 13).

$$w(x, y) = (x/L)^m \sin(\pi x/L)^2 (y/b_f) \quad (10)$$

$$\sigma_{cr,f,pin} = Et_f^2 [\alpha_4 L^2 (1 - \nu) + \alpha_5 b_f^2] / [100 L^2 b_f^2 (1 - \nu^2)] \quad (11)$$

$$\alpha_{4,PL} = 110 (b_f/L)^{0.2} \quad \alpha_{4,UDL} = 100 (b_f/L)^{0.2} \quad (12a, 12b)$$

$$\alpha_{5,PL} = 6\,500 (b_f/L)^{0.8} \quad \alpha_{5,UDL} = 7\,500 (b_f/L)^{0.8} \quad (13a, 13b)$$

Figure 3. Equation parameters α_1 , α_2 , and α_3 against L/b_f ratio. a) α_1 parameter, b) α_2 parameter, c) α_3 parameter.

3 THE ELASTIC LOCAL BUCKLING OF SIMPLY SUPPORTED BEAM'S WEB

The models of web used were of the following parameters: L of 600, 1000, 1500 and 2000 mm, h_w of 100, 200 and 400 mm, t_w of 2.2 and 5.2 mm for h_w of 100 and 400 mm, t_w of 2.2, 2.7, 3.2, 3.7, 4.2, 4.7 and 5.2 mm for h_w of 50 mm. For all the models the A_f was varying between 0, 50, 100, 200, 250, 300, 400, 500, 1 000, 2 000, 4 000, 5 000, 7 500 and 10 000 mm².

The influence of the flange onto the behaviour of the web is not limited to BC only. For the significant flange area A_f the additional shear stress caused by tension/compression pair of forces in flanges influences the critical buckling strength of the web.

The method proposed is similar to the one presented in preceding section. The following solution does not consider the influence of additional stress caused by tension/compression in flanges. $\sigma_{cr,w,fix,ns}$ stands for the elastic strength of the fixed web with shear flow disregarded.

$$w(x, z) = (x/L)^m \sin(n\pi x/L) \sin(\pi x/L)^2 (z^2/b_f) \sin(\pi z/b_f)^2 \quad (14)$$

$$\sigma_{cr,w,fix,ns} = Et_w^2 (\beta_1 L^4 + \beta_2 L^2 h_w^2 + \beta_3 h_w^4) / [1\ 000 L^2 h_w^4 (1 - \nu^2)] \quad (15)$$

$$\beta_{1,PL} = 13\ 500 (h_w/L)^2 \quad \beta_{1,UDL} = 9\ 500 (h_w/L)^2 \quad (16a, 16b)$$

$$\beta_{2,PL} = 10\ 000 + 25\ 000 (h_w/L)^2 \quad \beta_{2,UDL} = 9\ 000 + 22\ 500 (h_w/L)^2 \quad (17a, 17b)$$

$$\beta_{3,PL} = 37\ 500 + 9\ 500 (L/h_w)^2 \quad \beta_{3,UDL} = 23\ 500 + 6\ 000 (L/h_w)^2 \quad (18a, 18b)$$

The solution for the pinned web takes a similar form to the solution for the fixed web. Analyses of the elastic buckling strength of models with various BC and loading conditions lead to proposal of the reduction factor, χ , which can be applied to the Equation (15) in order to obtain the elastic buckling strength of the pinned web. Equations (19-21) propose the χ factor and the resultant $\sigma_{cr,w,pin,ns}$, critical elastic strength of the pinned web with shear flow disregarded.

$$\chi_{PL} = 1.55 + 0.08 (L/h_w)^{0.3} - 550 (t_w/h_w)^3 \quad (19)$$

$$\chi_{UDL} = 2.65 - 0.10 (L/h_w)^{0.3} - 1\ 250 (t_w/h_w)^3 \quad (20)$$

$$\sigma_{cr,w,pin,ns} = \sigma_{cr,w,fix,ns} / \chi \quad (21)$$

The consideration of the shear flow is based upon the assumption that the buckling mode can be either a bending buckling, coupled buckling or a shear buckling. In case of web with no shear flow the bending buckling takes place, changing to different forms as the A_f rises. (22) shows the relation between the maximal bending and shear stress in the web with no shear flow, ϑ_{ns} . Similarly, ϑ is a bending stress to shear stress relation with shear flow considered (23). The shear flow reduction factor ψ is proposed based upon the limitation of bending stress (in the form of upper limitation of 1) and balance between shear stress and shear flow dictated by aspect ratio dependent parameters α_ψ and β_ψ , see (24-26). (27) shall be used for determination of the buckling strength of the web for both PL and UDL loading scenarios.

$$\vartheta_{ns,PL} = h_w / (4L) \quad \vartheta_{ns,UDL} = h_w / (2L) \quad (22a, 22b)$$

$$\vartheta_{PL} = (A_f / t_w + h_w / 4) / L \quad \vartheta_{UDL} = (2A_f / t_w + h_w / 2) / L \quad (23a, 23b)$$

$$\psi = \min \left(1; \left[\alpha_\psi + \beta_\psi (\vartheta / \vartheta_{ns}) \right] \right) \quad (24)$$

$$\psi_{PL} = \min(1; \langle 0.7\sqrt{h_w/L} + 0.7(L/h_w)\{h_w/[4(A_f/t_w + h_w/4)]\} \rangle) \quad (25)$$

$$\psi_{UDL} = \min(1; \langle 0.7\sqrt{h_w/L} + 0.7(L/h_w)\{h_w/[2(A_f/t_w + h_w/2)]\} \rangle) \quad (26)$$

$$\sigma_{cr,w,fix} = \sigma_{cr,w,fix,ns} * \psi \quad \sigma_{cr,w,pin} = \sigma_{cr,w,pin,ns} * \psi \quad (27a, 27b)$$

Results from FEA (markers) and proposed equations (lines) are shown in Figure 4 for the 1000/200/2.2 PL series.

4 THE ELASTIC LOCAL BUCKLING CAPACITY OF SIMPLY SUPPORTED BEAM

The procedure of strength approximation of semi-fixed plate (see Figure 2) is as given. The plate is assumed to be restrained along long edges with a rotational spring. Flange models are restrained with a spring of K_w – stiffness corresponding to web stiffness per unit length – whilst web models are restrained with K_f (28). Based on the relation between the stiffnesses of plate and the restraint, the fixity factors for flange and web (γ_f and γ_w) are proposed (29 and 30). As γ varies between 0.0 and 1.0, the buckling mode of the segment varies between the pinned and fixed eigenform. The segment's capacity is as in (31).

Final equation (32) estimates the buckling capacity of the whole beam. The results of (31) are compared and the lower-bound value is assumed as the buckling strength of the whole beam. Figure 5 presents the comparison of (31 and 32) against results of FEA. Models of the

Figure 4. The strength reduction due to the shear flow effects. a) pinned web, b) fixed web.

Figure 5. Comparison of results from proposed equations (EQU) and numerical analyses (FEA). a) PL: σ_{cr} of the flange, web and beam (EQU) against σ_{cr} of the beam (FEA), b) UDL: σ_{cr} of the flange, web and beam (EQU) against σ_{cr} of the beam (FEA).

beams were built of segments having the same dimensions as in sections 2 and 3. All models were consisting of S4R5 elements (12 for each flange, 22 for web, 180 length-wise) of certain thickness representing beam segments made of elastic material, with the BC and loads as shown in Figure 1. The FEA software used for the analyses was SOFiSTiK, with a cross-check using Abaqus.

$$K_w = Et_w^3 / (2h_w) \quad K_f = Et_f^3 / (6b_f) \quad (28a, 28b)$$

$$\gamma_f = 0 < 1.15 * \arctan\{1.75 * [-0.25 - \log(K_f/K_w)]\} / \pi + 0.5 < 1 \quad (29)$$

$$\gamma_w = 0 < 1.15 * \arctan\{1.75 * [-0.25 + \log(K_f/K_w)]\} / \pi + 0.5 < 1 \quad (30)$$

$$\sigma_{cr,i} = \sigma_{cr,i,pin} + (\sigma_{cr,i,fix} - \sigma_{cr,i,pin}) * \gamma_i \text{ for } i \in (f; w) \quad (31)$$

$$\sigma_{cr,b} = \min(\sigma_{cr,f}; \sigma_{cr,w}) \quad (32)$$

5 SUMMARY

This research is aimed to explain the elastic buckling behaviour of simply supported beam by fulfilling 2 targets: investigate the strength of the segments of the beam considering in-between interactions and to propose equation estimating the local buckling capacity of the whole beam.

It can be stated, that equations based upon the energy equilibrium are feasible for the estimation of the elastic buckling capacity of the simple plates building up the beam. It was proven that, by utilising equations for the estimation of the effects of the web-flange interactions, it is possible to approximate the elastic local buckling capacity of the beam with satisfying precision.

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